Int. J. Multiphase Flow, Vol. 5, pp. 233–242
Pergamon/Elsevier, 1979. Printed in Great Britain

THE ANALYSIS OF VIRTUAL MASS EFFECTS IN TWO-PHASE FLOW

D. DREW, L. CHENG and R. T. LAHEY, JR.
Rensselaer Polytechnic Institute, Troy, NY 12181, U.S.A.

(*Received* 15 *August* 1978; *in revised form* 15 *March* 1979)

Abstract—One of the primary difficulties in the mathematical modelling of two-phase flows is the complexity of the interfacial transfer phenomena. The present study is concerned with the so-called virtual mass force during the acceleration of a two-phase mixture. It is shown that this interfacial force must be objective, and thus invariant under a change of reference frame. The most general form of an objective virtual mass acceleration is derived and appropriate experiments are suggested for verification and parameter determination.

1. INTRODUCTION

One of the outstanding problems in two-phase flow is concerned with phase separation mechanisms. This paper focuses on understanding the effect of virtual mass on phase separation during the acceleration of a two-phase mixture.

A bubble imbedded in a flowing fluid is influenced by a number of mechanisms (drag force, virtual mass force, Bassett force, etc.), which act on it through the traction at the vapor–liquid interface. The traction at the interface depends on the geometry of the interface, and the details of the flow field inside and outside of the bubble. It is not practical to solve the exact fluid equation (plus jump conditions) for two-phase flows of practical significance. Hence to describe multidimensional two-phase flows, one must resort to averaged equations, such as the time averaged equations of Ishii (1975). The basic premise is that not all of the details of the exact flow equations contribute to the *averaged* traction at the interface. The philosophy adopted for the determination of phenomenological constitutive equations is that the interfacial force term can be related to some appropriate set of flow variables. For example, the interfacial force term is usually related to the average relative velocity through an interfacial drag law. This can be done by assuming that the interfacial drag force is proportional to the (vectorial) relative velocity, and using experimental data to infer the dependence of the proportionality factor on the void fraction and the magnitude of the relative velocity. We have assumed that the virtual mass force should be treated in a similar way. That is, we have assumed that the virtual mass force is proportional to an appropriate acceleration. As before, the proportionality factor must be determined experimentally. Introducing the virtual mass force in this manner should yield a working system of conservation equations which are able to model accelerating two-phase flows better than a comparable system of equations without a virtual mass force term.

The basic concept of a virtual mass force can be easily understood by considering the change in kinetic energy of fluid surrounding an accelerating sphere. The classical result, contained in the works of Milne-Thomson (1968), is that the acceleration of the sphere induces a resisting force on the sphere equal to one-half the mass of the displaced fluid times the acceleration of the sphere. That is, if $u(t)$ is the velocity of the center-of-mass of the sphere, ρ_L is the density of the liquid, and R is the radius of the sphere, then the virtual (or added) mass force on the sphere is defined as:

$$-\frac{2}{3}\pi R^3 \rho_L \frac{\mathrm{d}u}{\mathrm{d}t}. \qquad [1]$$

In arriving at this expression for the virtual mass force, the effects of any viscous forces and neighboring spheres were not considered. In general, the virtual mass force for a real flow, involving many interacting bubbles, must include other flow parameters such as void fraction.

There are other forces, e.g. the Bassett force, the Faxén forces, the lift force, the Magnus force, and forces due to deformation/expansion/contraction of the bubbles. Here we shall be concerned only with the virtual mass force.

2. THE MOMENTUM EQUATIONS OF SEPARATED FLOW

Consider the phasic axial momentum equations of transient two-phase flow (Ishii 1975),

$$\alpha_k \rho_k \frac{D_k \mathbf{u}_k}{Dt} = -\alpha_k \nabla p_k + \nabla \cdot [\alpha_k(\underset{\sim}{\tau}_k + \underset{\sim}{\tau}_k{}^T)] + \alpha_k \rho_k \mathbf{g}_k + (p_{ki} - p_k)\nabla \alpha_k + (\mathbf{u}_{ki} - \mathbf{u}_k)\Gamma_k + \mathbf{M}_k \qquad [2]$$

where the subscript k denotes the liquid ($k = L$) and vapor ($k = G$) phases; and where α_k is the volume fraction of phase k, p is the pressure, $\underset{\sim}{\tau}$ is the laminar stress, $\underset{\sim}{\tau}^T$ is the turbulent stress, g is the body force and D_k/Dt is the material derivative following phase k.

For adiabatic air/water flow, the volumetric mass generation term for phase-k Γ_k is zero. Furthermore, if the phasic pressure p_k and the interfacial pressure p_{ki} are assumed identical:

$$\alpha_k \rho_k \frac{D_k \mathbf{u}_k}{Dt} = -\alpha_k \nabla p_k + \nabla \cdot [\alpha_k(\underset{\sim}{\tau}_k + \underset{\sim}{\tau}_k{}^T)] + \alpha_k \rho_k \mathbf{g}_k + \mathbf{M}_k. \qquad [3]$$

The volumetric interfacial force $\mathbf{M}_k$ on phase-k is the force on one phase due to the other phase. For adiabatic air/water flow, $\mathbf{M}_k$ contains the interfacial drag force ($\mathbf{F}_{D_i}$) and the virtual mass force ($\mathbf{F}_{VM}$). For the purposes of this discussion only the virtual mass force per unit bubble (vapor phase) volume will be considered:

$$\mathbf{F}_{VM} = C_{VM} \rho_L \mathbf{a}_{VM} \qquad [4]$$

where $\mathbf{a}_{VM}$ is the virtual mass acceleration term in question, and $C_{VM}\rho_L$ is the virtual mass per unit bubble volume. As discussed previously, for a single, non-deformable, spherical particle,

$$C_{VM} = 1/2. \qquad [5]$$

For two-phase flows of practical concern, the appropriate void dependent expression for C_{VM} is likely to give a value less than one-half, but its functional form is currently not well known. (A more general theory would allow C_{VM} to be a second order tensor. If the two-phase flow is locally isotropic, the more general theory reduces to [4]).

Constitutively the virtual mass force can be treated as a product of the virtual mass and virtual mass acceleration. Constitutive equations, such as that for interfacial drag and interfacial stress tensors, are invariant under changes of frame of reference, i.e. they are objective. The starting hypothesis in formulating a constitutive relationship for the virtual mass force is that the virtual mass acceleration term ($\mathbf{a}_{VM}$) should be objective, that is, it should be frame indifferent.

3. OBJECTIVITY

"A frame of reference may be described as a possible way of relating physical reality to a three-dimensional Euclidean point space and a real time axis" (Truesdell & Noll 1965). A change of frame is a one-to-one mapping of space-time onto itself in such a manner that distances, time intervals, and temporal order remain unchanged.

To better understand the meaning of objectivity, consider its mathematical definition. Let x and t denote the position and time in the old frame, $\mathbf{x}^*$ and t^* the corresponding position and time in the new frame. The most general change of frame is of the form,

$$\mathbf{x}^*(t) = \mathbf{Y}(t) + \underset{\sim}{\mathbf{Q}}(t) \cdot (\mathbf{x}(t) - \mathbf{z}_0) \tag{6}$$

$$t^* = t + t' \tag{7}$$

where the quantity t' is a real number.

Without loss of generality, let frame-I move relative to frame-II in a rigid body motion. As shown schematically in figure 1, a point P is observed in frame-I at position x, is observed in frame-II at position $\mathbf{x}^*$.

The fixed point $\mathbf{z}_0$ is mapped into $Y(t)$. If $\mathbf{z}_0$ is set to zero, then $\mathbf{Y}(t)$ represents the position vector of the origin of frame-I with respect to the new frame-II. In other words, $\mathbf{Y}(t)$ represents a translation. The time dependent orthogonal† second-order tensor $\underset{\sim}{\mathbf{Q}}(t)$ represents a rigid body rotation and possibly a reflection; $\det[\underset{\sim}{\mathbf{Q}}] = 1$ implies a rotation, while $\det[\underset{\sim}{\mathbf{Q}}] = -1$ signifies a reflection.

A scalar is objective if,

$$\alpha^* = \alpha. \tag{8}$$

A vector $\mathbf{u}$ is objective if,

$$\mathbf{u}^* = \underset{\sim}{\mathbf{Q}} \cdot \mathbf{u}. \tag{9}$$

A second order tensor $\underset{=}{\mathbf{T}}$ is objective if,

$$\underset{=}{\mathbf{T}}^* = \underset{\sim}{\mathbf{Q}} \cdot \underset{=}{\mathbf{T}} \cdot \underset{\sim}{\mathbf{Q}}^T. \tag{10}$$

Not all vectors and tensors are objective. Indeed, only very special ones are. As an example, related to two-phase flow, consider a position vector x in the new frame,

$$\mathbf{x}^*(t) = \mathbf{Y}(t) + \underset{\sim}{\mathbf{Q}} \cdot (\mathbf{x}(t) - z_0). \tag{11}$$

The material derivative of [11],

$$\frac{D_k \mathbf{x}^*}{Dt} = \dot{\mathbf{Y}} + \underset{\sim}{\mathbf{Q}} \cdot \left(\frac{D_k \mathbf{x}}{Dt}\right) + \dot{\underset{\sim}{\mathbf{Q}}} \cdot (\mathbf{x} - \mathbf{z}_0) \tag{12}$$

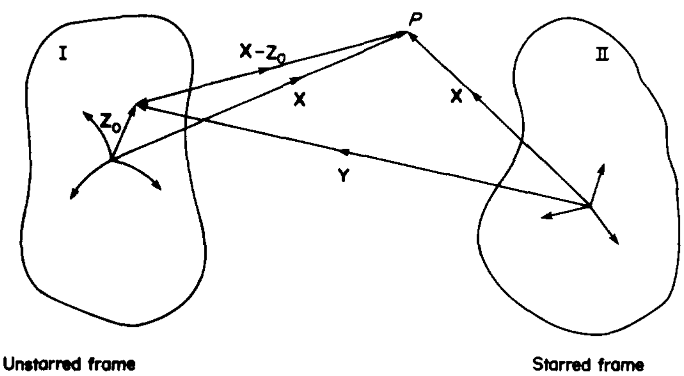

Figure 1. Change of reference frame.

†By orthogonal we mean $\underset{\sim}{\mathbf{Q}} \cdot \underset{\sim}{\mathbf{Q}}^T = \underset{=}{\mathbf{I}}$.

where

$$\frac{D_k}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}_k \cdot \nabla. \quad [13]$$

By definition, the velocity of phase-k is given by,

$$\mathbf{u}_k = \frac{D_k \mathbf{x}}{Dt}. \quad [14]$$

Thus, [12] and [14] yield,

$$\overset{*}{\mathbf{u}}_k = \mathbf{Q} \cdot \mathbf{u}_k + \dot{\mathbf{Q}} \cdot (\mathbf{x} - \mathbf{z}_0) + \dot{\mathbf{Y}}. \quad [15]$$

Comparing [9] and [15], it is obvious that phasic velocities are not objective. If, however the liquid phase velocity ($k = L$) is subtracted from the vapor phase velocity ($k = G$):

$$\mathbf{u}_G^* - \mathbf{u}_L^* = \mathbf{Q} \cdot (\mathbf{u}_G - \mathbf{u}_L). \quad [16]$$

Hence, the relative velocity is objective.

The variables used in [3] are all time averaged variables. That is,

$$f(t_0) = \frac{1}{T} \int_{t_0}^{t_0+T} \tilde{f} \, \mathrm{d}t \quad [17]$$

where $\tilde{f}$ is the instantaneous value of the variable and f is the time averaged value. The time average of $\tilde{f}$ depends on the time *interval* over which the integration occurs. Thus, the time averaging process, [17], is objective.

One of the cornerstones of classical mechanics is that the surface traction $(\mathbf{n} \cdot \underset{\sim}{\tau})$ is objective (Truesdell & Toupin 1960). For a two-phase system, this implies that the time averaged interfacial traction, $(\mathbf{n} \cdot \underset{\sim}{\tau})|_i$, is objective. Since Ishii (1975) has shown,

$$\mathbf{n} \cdot \underset{\sim}{\tau}|_i = p \nabla \alpha + \mathbf{M}_k \quad [18]$$

$\mathbf{M}_k$ must be objective if $p \nabla \alpha$ is objective. It is relatively straightforward to show (Cheng 1977) that,

$$p^* \nabla^* \alpha^* = \mathbf{Q} \cdot (p \nabla \alpha). \quad [19]$$

Thus, $p \nabla \alpha$ is objective and hence the volumetric interfacial force, $\mathbf{M}_k$, must be objective. That is, quantities which express how the two-phase mixture reacts to a given (geometric, thermal and mechanical) state must be coordinate frame invariant. Specifically, laws which express the volumetric interfacial force, $\mathbf{M}_k$, as a function of mean field variables, must be the same in any reference frame. Thus, the only admissable interfacial forces must transform as,

$$\mathbf{M}_k^* = \mathbf{Q} \cdot \mathbf{M}_k. \quad [20]$$

4. OBJECTIVE VIRTUAL MASS ACCELERATIONS

For the case of adiabatic air/water flow, the interfacial force per unit mixture volume, $\mathbf{M}_k$, is comprised of the interfacial drag force per unit bubble volume ($\mathbf{F}_{D_i}$), and the virtual mass force per unit bubble volume ($\mathbf{F}_{VM}$). Thus,

$$\mathbf{M}_L = \mathbf{M}_G = \alpha[\mathbf{F}_{D_i} + \mathbf{F}_{VM}]. \quad [21]$$

Since the drag force involves only relative velocities, it is obviously objective; thus determine the objective virtual mass accelerations, $\mathbf{a}_{VM}$.

The phasic acceleration terms:

$$\frac{D_k\mathbf{u}_k^*}{Dt} = \underset{\sim}{\mathbf{Q}} \cdot \frac{D_k\mathbf{u}_k}{Dt} + 2\dot{\underset{\sim}{\mathbf{Q}}} \cdot \mathbf{u}_k + \ddot{\underset{\sim}{\mathbf{Q}}} \cdot (\mathbf{x} - \mathbf{z}_0) + \ddot{\mathbf{Y}}. \quad [22]$$

Subtracting this equation for the liquid phase ($k = L$), from the equation for the vapor phase ($k = G$):

$$\frac{D_G\mathbf{u}_G^*}{Dt} - \frac{D_L\mathbf{u}_L^*}{Dt} = \underset{\sim}{\mathbf{Q}} \cdot \left[\frac{D_G\mathbf{u}_G}{Dt} - \frac{D_L\mathbf{u}_L}{Dt}\right] + 2\dot{\underset{\sim}{\mathbf{Q}}} \cdot [\mathbf{u}_G - \mathbf{u}_L]. \quad [23]$$

To find an expression for $\dot{\underset{\sim}{\mathbf{Q}}}$ consider the velocity of phase-k in indical notation. Taking the material derivative of [6]:

$$u_{ki}^* = \dot{Y}_i + \dot{Q}_{ij}(x_j - Z_0) + Q_{ij}u_{kj} \quad [24]$$

where

$$u_{ki}^* = \frac{D_k x_i^*}{Dt} \quad [25]$$

and

$$u_{kj} = \frac{D_k x_j}{Dt}. \quad [26]$$

The velocity gradient of phase-k in the starred frame of reference is,

$$\frac{\partial u_{kj}}{\partial x_l^*} = \frac{\partial u_{kj} \partial x_m}{\partial x_m \partial x_l^*}. \quad [27]$$

From the definition of the rotation tensor $\underset{\sim}{\mathbf{Q}}$:

$$\frac{\partial x_m}{\partial x_l^*} = Q_{lm}. \quad [28]$$

Substituting [28] into [27]:

$$\frac{\partial u_{kj}}{\partial x_l^*} = Q_{lm} \frac{\partial u_{kj}}{\partial x_m}. \quad [29]$$

The gradient, $\partial/\partial x_l^*$ of [24] is,

$$\frac{\partial u_{ki}^*}{\partial x_l^*} = \dot{Q}_{ij} \frac{\partial x_j}{\partial x_l^*} + Q_{ij} \frac{\partial u_{kj}}{\partial x_l^*}. \quad [30]$$

Using the result from [28]:

$$\frac{\partial x_j}{\partial x_l^*} = Q_{lj}. \qquad [31]$$

Rearranging [30] with results from [29] and [31]:

$$\frac{\partial u_{ki}^*}{\partial x_l^*} = \dot{Q}_{ij}Q_{lj} + Q_{ij}Q_{lm}\frac{\partial u_{kj}}{\partial x_m}. \qquad [32]$$

When [32] is written in invariant notation:

$$\nabla^*\mathbf{u}_k^* = \underset{\sim}{\mathbf{Q}} \cdot \underset{\sim}{\dot{\mathbf{Q}}}^T + \underset{\sim}{\mathbf{Q}} \cdot (\nabla\mathbf{u}_k) \cdot \underset{\sim}{\mathbf{Q}}^T. \qquad [33]$$

Dotting [33] with $\underset{\sim}{\mathbf{Q}}^T$ yields,

$$\underset{\sim}{\mathbf{Q}}^T \cdot (\nabla^*\mathbf{u}_k^*) = \underset{\sim}{\dot{\mathbf{Q}}}^T + (\nabla\mathbf{u}_k) \cdot \underset{\sim}{\mathbf{Q}}^T \qquad [34]$$

or,

$$\underset{\sim}{\dot{\mathbf{Q}}}^T = \underset{\sim}{\mathbf{Q}}^T \cdot (\nabla^*\mathbf{u}_k^*) - (\nabla\mathbf{u}_k) \cdot \underset{\sim}{\mathbf{Q}}^T. \qquad [35]$$

By transposing [35] we arrive at the final form for $\underset{\sim}{\dot{\mathbf{Q}}}$,

$$\underset{\sim}{\dot{\mathbf{Q}}} = (\nabla^*\mathbf{u}_k^*)^T \cdot \underset{\sim}{\mathbf{Q}} - \underset{\sim}{\mathbf{Q}} \cdot (\nabla\mathbf{u}_k)^T. \qquad [36]$$

Writing [36] for the vapor phase ($k = G$) and the liquid phase ($k = L$) separately,

$$\underset{\sim}{\dot{\mathbf{Q}}} = (\nabla^*\mathbf{u}_G^*)^T \cdot \underset{\sim}{\mathbf{Q}} - \underset{\sim}{\mathbf{Q}} \cdot (\nabla\mathbf{u}_G)^T \qquad [37]$$

and,

$$\underset{\sim}{\dot{\mathbf{Q}}} = (\nabla^*\mathbf{u}_L^*)^T \cdot \underset{\sim}{\mathbf{Q}} - \underset{\sim}{\mathbf{Q}} \cdot (\nabla\mathbf{u}_L)^T. \qquad [38]$$

If we write $2\underset{\sim}{\dot{\mathbf{Q}}}$ as

$$2\underset{\sim}{\dot{\mathbf{Q}}} = (2-\lambda)\underset{\sim}{\dot{\mathbf{Q}}} + \lambda\underset{\sim}{\dot{\mathbf{Q}}}, \qquad [39]$$

where λ is a parameter and combine [37]–[39]:

$$\underset{\sim}{\dot{\mathbf{Q}}} = (2-\lambda)[(\nabla^*\mathbf{u}_G^*)^T \cdot \underset{\sim}{\mathbf{Q}} - \underset{\sim}{\mathbf{Q}} \cdot (\nabla\mathbf{u}_G)^T] + \lambda[(\nabla^*\mathbf{u}_L^*)^T \cdot \underset{\sim}{\mathbf{Q}} - \underset{\sim}{\mathbf{Q}} \cdot (\nabla\mathbf{u}_L)^T]. \qquad [40]$$

Substituting [40] into [23], after some simplification:

$$\frac{D_G\mathbf{u}_G^*}{Dt} - \frac{D_L\mathbf{u}_L^*}{Dt} - (\mathbf{u}_G^* - \mathbf{u}_L^*) \cdot \nabla^*\mathbf{u}_G^* - (\mathbf{u}_G^* - \mathbf{u}_L^*) \cdot \nabla^*\mathbf{u}_L^* + (1-\lambda)[(\mathbf{u}_G^* - \mathbf{u}_L^*) \cdot \nabla^*(\mathbf{u}_L^* - \mathbf{u}_G^*)]$$

$$= \underset{\sim}{\mathbf{Q}} \cdot \left\{\frac{D_G\mathbf{u}_G}{Dt} - \frac{D_L\mathbf{u}_L}{Dt} - (\mathbf{u}_G - \mathbf{u}_L) \cdot \nabla\mathbf{u}_G - (\mathbf{u}_G - \mathbf{u}_L) \cdot \nabla\mathbf{u}_L + (1-\lambda)[(\mathbf{u}_G - \mathbf{u}_L) \cdot \nabla(\mathbf{u} - \mathbf{u}_G)]\right\}. \qquad [41]$$

Observe that the relative acceleration given in [41] is objective. After rearranging the acceleration term in [41] we get the most general objective acceleration ($\mathbf{a}_{VM}$) for two phase flow,

$$\mathbf{a}_{VM}=\left(\frac{D_G\mathbf{u}_G}{Dt}-(\mathbf{u}_G-\mathbf{u}_L)\cdot\nabla\mathbf{u}_G\right)-\left(\frac{D_L\mathbf{u}_L}{Dt}+(\mathbf{u}_G-\mathbf{u}_L)\cdot\nabla\mathbf{u}_L\right)+(1-\lambda)[(\mathbf{u}_G-\mathbf{u}_L)\cdot\nabla(\mathbf{u}_L-\mathbf{u}_G)]. \quad [42]$$

Equation [42] can be written in several equivalent forms, e.g.

$$\mathbf{a}_{VM}=\frac{D_L\mathbf{u}_G}{Dt}-\frac{D_G\mathbf{u}_L}{Dt}+(1-\lambda)(\mathbf{u}_G-\mathbf{u}_L)\cdot\nabla(\mathbf{u}_L-\mathbf{u}_G) \quad [43]$$

or,

$$\mathbf{a}_{VM}=\frac{D_G(\mathbf{u}_G-\mathbf{u}_L)}{Dt}+(\mathbf{u}_G-\mathbf{u}_L)\cdot[(\lambda-2)\nabla\mathbf{u}_G+(1-\lambda)\nabla\mathbf{u}_L]. \quad [44]$$

The first term in [44] $D_G(\mathbf{u}_G-\mathbf{u}_L)/Dt$, has been used as the virtual mass acceleration term by previous investigators, Wallis (1969), Hinze (1961) and Soo (1967). Equation [44] clearly shows that this term, alone, is not objective and is thus not a possible choice for $\mathbf{a}_{VM}$.

5. EVALUATION OF THE PARAMETERS C_{VM} AND λ

Consider the parameters C_{VM} and λ. We anticipate that $C_{VM}=C_{VM}(\alpha)$ and $\lambda=\lambda(\alpha)$. From [4] and [44],

$$\mathbf{F}_{VM}=\rho_L C_{VM}\left\{\frac{\partial(\mathbf{u}_G-\mathbf{u}_L)}{\partial t}+\mathbf{u}_G\cdot\nabla(\mathbf{u}_G-\mathbf{u}_L)+(\mathbf{u}_G-\mathbf{u}_L)\cdot[(\lambda-2)\nabla\mathbf{u}_G+(1-\lambda)\nabla u_L]\right\}. \quad [45]$$

Note in [45] that the appropriate values for C_{VM} can be determined from experiments involving only the temporal acceleration term, which is given by,

$$\rho_L C_{VM}\frac{\partial(\mathbf{u}_G-\mathbf{u}_L)}{\partial t}. \quad [46]$$

Hence, to determine C_{VM}, we can perform an experiment in which,

$$\mathbf{u}_G=u_G(t)\mathbf{i} \quad [47a]$$

$$\mathbf{u}_L=u_L(t)\mathbf{i} \quad [47b]$$

where $\mathbf{i}$ is a unit vector in the axial direction. Houghton (1976) has performed such experiments with single dispersed particles of various shapes. He found, as expected, that for a sphere, $C_{VM}=\frac{1}{2}$, and that for other shapes, the value of C_{VM} is shape dependent. These experimental values are considered to be valid for dilute two-phase mixtures. For higher void fraction cases of more practical concern, the interaction between the dispersed particles, which likely lowers the value of C_{VM}, is not well known. Obviously, more experimental data are needed. Consider some experiments which might be performed to determine the correct value of the parameter λ.

Consider the case of the acceleration of a rigid single spherical bubble (released from rest) through a still tank. For this case, the time averaged variables $\mathbf{u}_L$ and $\mathbf{u}_G$ are,

$$\mathbf{u}_G = u_G(\mathbf{x}, t)\mathbf{i}$$
$$\mathbf{u}_L = 0. \qquad [48]$$

Thus, [45] reduces to,

$$\mathbf{F}_{VM} = \frac{1}{2}\rho_L\left\{\frac{D_G\mathbf{u}_G}{Dt} + (\lambda - 2)\mathbf{u}_G \cdot \nabla\mathbf{u}_G\right\}. \qquad [49]$$

This particular problem (a bubble rising in a still tank) is a classical one, and has a well known formulation,

$$\mathbf{F}_{VM} = \frac{1}{2}\rho_L\frac{D_G\mathbf{u}_G}{Dt}. \qquad [50]$$

Comparing [49] and [50], we find that for this case, which is typical of very low void fraction, $\lambda = 2$.

Consider very high void fraction, and the virtual mass force ($\mathbf{F}_{VM}$) on dispersed liquid droplets. For a single liquid droplet, accelerating in a constant velocity (vapor) flow field, the time averaged velocities are,

$$\mathbf{u}_G = \text{constant}$$
$$\mathbf{u}_L = u_L(\mathbf{x}, t)\mathbf{i}. \qquad [51]$$

Thus, using [42] and [4], the virtual mass force on an accelerating spherical droplet can be written as,

$$\mathbf{F}_{VM} = \rho_L C_{VM}\left\{\frac{D_L\mathbf{u}_L}{Dt} + (\mathbf{u}_G - \mathbf{u}_L)\cdot\nabla\mathbf{u}_L - (1-\lambda)(\mathbf{u}_G - \mathbf{u}_L)\cdot\nabla\mathbf{u}_L\right\}. \qquad [52]$$

This problem is also a classical one, which, for a spherical droplet, has a well known formulation,

$$\mathbf{F}_{VM} = -\frac{1}{2}\rho_G\frac{D_L\mathbf{u}_L}{Dt}. \qquad [53]$$

By comparing [52] and [53], we find that $\lambda = 0$, and $C_{VM} = \frac{1}{2}\rho_G/\rho_L$.

In summary, we have shown through comparisons with the asymptotic cases of single bubble, and single droplet, acceleration that the parameter λ should be a function of (at least) void fraction. Moreover, it appears that the limiting values of λ in the low void and high void regimes are,

$$\lim_{\alpha\to 0}\lambda(\alpha) = 2 \qquad [54a]$$
$$\lim_{\alpha\to 1}\lambda(\alpha) = 0. \qquad [54b]$$

The value of $\lambda(\alpha)$ at intermediate void fractions is not clear, and must be determined experimentally. With this in mind, let us consider some appropriate experiments which could be used to verify the validity of [44], and to determine the appropriate functional form of $\lambda(\alpha)$.

6. POSSIBLE EXPERIMENTS

From [44], and the previous discussion, it should be clear that an appropriate experiment must be one in which,

$$\mathbf{u}_G - \mathbf{u}_L \neq 0 \qquad [55a]$$

$$[\mathbf{u}_G - \mathbf{u}_L] \cdot \nabla \mathbf{u}_G \neq 0 \qquad [55b]$$

$$[\mathbf{u}_G - \mathbf{u}_L] \cdot \nabla \mathbf{u}_L \neq 0. \qquad [55c]$$

It is also interesting to note that it is the spatial acceleration terms which are of interest, since the temporal acceleration term in [44] is the same as that used by previous investigators, Wallis (1969), Hinze (1961) and Soo (1967).

An experiment which appears to be attractive is one in which there is a spatial acceleration, or deceleration, of a two-phase flow. That is, a nozzle/diffuser experiment, such as Wallis (1977). It clearly satisfies the criteria given in [55], unfortunately, due to the relatively small spatial accelerations which can be measured with optical techniques, it can be readily shown (Cheng *et al.* 1978) that such experiments do not yield much insight into the form of $C_{VM}(\alpha)$ and $\lambda(\alpha)$. Indeed, it appears that virtual mass effects are significant only for conditions were spatial effects are important, such as critical two-phase flows.

Thus the most promising experiments for the verification of [44], and the determination of $C_{VM}(\alpha)$ and $\lambda(\alpha)$, appear to be air/water critical flows or ones in which the sonic wave dispersion characteristics are measured. It is hoped that future experiments of this type will be performed.

7. APPLICATION

While, for many cases of interest, the inclusion, or neglect, of the virtual mass force in the phasic momentum equations does not appreciably change the numerical results, the computation efficiency of the solution scheme can be dramatically effected. This effect is due to the fact that the virtual mass force changes the eigenvalues of the system of partial differential equations being solved (Cheng *et al.* 1978).

For example, the system of six partial differential equations, one-dimensional phasic continuity, momentum and energy equations, can be written in matrix form as:

$$A\frac{\partial \mathbf{u}}{\partial t} + B\frac{\partial \mathbf{u}}{\partial z} + C\mathbf{u} = \mathbf{d}. \qquad [56]$$

Where $\mathbf{u}$ is the vector of the dependent variables and A, B and C are square matrices. We see from [4], [21] and [44], that the temporal and spatial derivative terms in the virtual mass force will appear in the A and B matrices. If we do not include the virtual mass force then these terms are absent. The classification of the system (i.e. elliptic, parabolic or hyperbolic) is determined from (Garabelian 1964),

$$\det[A\mu - B] = 0 \qquad [57]$$

where the eigenvalues, $\mu_i = dz_i/dt$, determine the characteristics of the system. Thus, inclusion of the virtual mass force term changes the eigenvalues, and can change the classification of the system. In general, inclusion of the virtual mass force dramatically improves the computation efficiency. See Lahey *et al.* (1978) for a discussion of this effect. Moreover, for conditions in which one has very high spatial accelerations, such as critical two-phase flow, the virtual mass force is of prime importance, and can not be neglected.

Acknowledgements—The authors gratefully acknowledge the financial support of this study by the USNRC-RSR and the stimulating technical discussions with Professor J. Flaherty (RPI), Dr. N. Zuber (USNRC) and Professor C. Truesdell (Johns Hopkins).

NOMENCLATURE

$\mathbf{a}_{VM}$ virtual mass acceleration
C_{VM} virtual volume
D_k/Dt material derivative following phase k
$\mathbf{F}_{D_i}$ interfacial drag force
$\mathbf{F}_{VM}$ virtual mass force
$\mathbf{g}_k$ body force on phase k
G subscript denoting gas (vapor) phase
k G or L
L subscript denoting liquid phase
$\mathbf{M}_k$ interfacial force density
p_k pressure of phase k
p_{k_i} interfacial pressure of phase k
$\mathbf{Q}$ orthonormal (rotation) matrix
$\tilde{R}$ bubble radius
t time
$\mathbf{u}_k$ velocity of phase k
$\mathbf{u}_{ki}$ interfacial velocity of phase k
$\mathbf{x}$ spatial coordinates

Greek symbols

α_k volume fraction of phase k
Γ_k interfacial mass generation rate to phase k
λ parameter in virtual mass acceleration
ρ_k density of phase k
τ_k laminar (viscous) stress of phase k
$\tau_k{}^T$ turbulent stress of phase k

REFERENCES

Cheng, L. Y., Drew, D. A. & Lahey, R. T., Jr. 1978 Virtual mass effects in two phase flow. NUREG/CR0020.

Garabedian, P. R. 1964 *Partial Differential Equations.* Wiley, New York.

Hinze, J. O. 1962 Momentum and mechanical-energy balance equations for a flowing homogeneous suspension with slip between the two phases. *Appl. Sci. Res.* Sec. A **11**, 33–46.

Houghton, G. 1976 Final report on subcontract (S-1101) for the investigation of particle-fluid forces in accelerative flows. Letter to V. Ransom (*EG&G*).

Ishii, M. 1975 *Thermo-Fluid Dynamic Theory of Two-Phase Flow.* Eyrolles, Paris.

Lahey, R. T., Jr., Cheng, L. Y., Drew, D. A. & Flaherty, J. E. 1978 The effect of virtual mass on the numerical stability of accelerating two-phase flows. *A.I.Ch.E. Paper No.* 56*b* 71st Annual Meeting, Miami Beach, Florida, 12–16 November 1978.

Milne-Thomson, L. M. 1968 *Theoretical Hydrodynamics*, 5th Edn. Macmillan, New York.

Soo, S. L. 1967 *Fluid Dynamics of Multiphase Systems.* Blaisdell Press, New York.

Truesdell, C. & Noll, W. 1965 *Handbuch der Physik*, Vol. III/3. Springer-Verlag, Berlin.

Truesdell, C. & Toupin, R. 1960 Classical Field Theories. *Handbuck der Physik*, Vol. III/1, Springer-Verlag, Berlin.

Wallis, G. B. 1969 *One Dimensional Two-Phase Flow.* McGraw-Hill, New York.

Wallis, G. B., Richter, H. J. & Kuo, J. T. 1967 The separated flow model of two-phase flow. *EPRI-NP*-275.